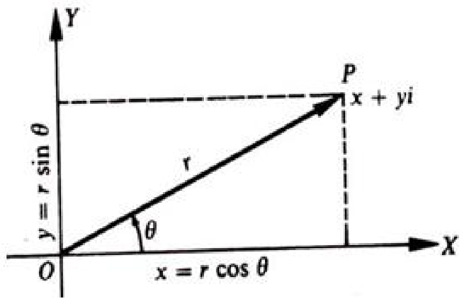
4.5

Polar Form

Learning Objectives:

- To represent a complex number in polar form and vice versa
- To learn multiplication and division of two complex numbers in polar form
  AND
- To practice the related problems

Let the complex number $x + iy$ be represented by the vector OP. This vector may be described in terms of the length r of the vector and any positive angle θ which the vector makes with the positive x-axis. The number $r = \sqrt{x^2 + y^2}$ is called the *modulus* or *absolute value* of the complex number and the angle θ is called the *amplitude* or *argument* of the complex number.



From the figure,

$$x = r\cos\theta \quad , \qquad y = r\sin\theta$$

Then

$$z = x + iy = r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta)$$

We call

$$\boldsymbol{z = r(\cos\theta + i\sin\theta)}$$

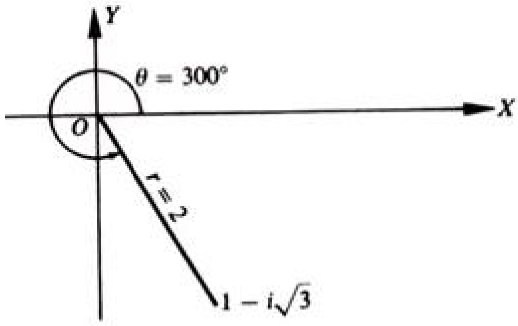
the *polar form* of z and

$$\boldsymbol{z = x + iy}$$

the *rectangular form* of the complex number z.

Example: Express $z = 1 - i\sqrt{3}$ in polar form.

Solution:



The modulus $r = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = 2$

$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

The amplitude θ is either $120°$ or $300°$. Now we know that P lies in quadrant IV, hence, $\theta = 300°$.

The required polar form is

$$z = r(\cos\theta + i\sin\theta) = 2(\cos 300° + i\sin 300°)$$

Example: Express the complex number $z = 8(\cos 210° + i\sin 210°)$ in rectangular form.

Solution:

Since $\cos 210° = -\frac{\sqrt{3}}{2}$, $\sin 210° = -\frac{1}{2}$

We have $z = 8(\cos 210° + i\sin 210°)$

$$= 8\left[-\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right] = -4\sqrt{3} - 4i$$

The required rectangular form is $z = -4\sqrt{3} - 4i$

Multiplication and Division in Polar Form

Multiplication

The modulus of the product of two complex numbers is the product of their moduli, and the amplitude of the product is the sum of their amplitudes.

We prove the above theorem as follows.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2\begin{bmatrix}(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\\ +i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\end{bmatrix}$$

$$= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

The theorem is proved.

Division

The modulus of the quotient of two complex numbers is the modulus of the dividend divided by the modulus of the divisor, and the amplitude of the quotient is the amplitude of the dividend minus the amplitude of the divisor.

We prove the above theorem as follows.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)} = \frac{r_1(\cos\theta_1+i\sin\theta_1)(\cos\theta_2-i\sin\theta_2)}{r_2(\cos\theta_2+i\sin\theta_2)(\cos\theta_2-i\sin\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\cdot\frac{(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2)}{\cos^2\theta_2+\sin^2\theta_2}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

The theorem is proved.

Example: Given $z_1 = 2(\cos 300° + i \sin 300°)$ and $z_2 = 8(\cos 210° + i \sin 210°)$, find

a) the product $z_1 z_2$

b) the quotient $\frac{z_1}{z_2}$

c) the quotient $\frac{z_2}{z_1}$

Solution:

(a) The modulus of the product is $2 \times 8 = 16$.

The amplitude is $300° + 210° = 510°$, but, following the convention, we shall use the smallest positive coterminal angle $510° - 360° = 150°$. Thus

$$z_1 z_2 = 16(\cos 150° + i \sin 150°)$$

(b) The modulus of the quotient $\frac{z_1}{z_2}$ is $2 \div 8 = \frac{1}{4}$.

The amplitude is $300° - 210° = 90°$. Thus

$\frac{z_1}{z_2} = \frac{1}{4}(\cos 90° + i \sin 90°)$

(c) The modulus of the quotient $\frac{z_2}{z_1}$ is $8 \div 2 = 4$.

The amplitude is $210° - 300° = -90°$, but we shall use the smallest positive coterminal angle $360° - 90° = 270°$. Thus

$\frac{z_2}{z_1} = 4(\cos 270° + i \sin 270°)$

P1:

Find the polar form of $\sqrt{3}+i$.

Solution:

Let $z = \sqrt{3} + i = r(cos\theta + isin\theta)$, where

$$r = |z| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2\,,$$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Notice that z lies in quadrant-I. So, $\theta = \frac{\pi}{6}$

∴ The Polar form of $\sqrt{3} + i$ is $2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$.

P2:

Find the rectangular form of $2\left(\cos 315^{\circ} + i \sin 315^{\circ}\right)$.

Solution:

$$2(\cos 315^{\circ} + i\sin 315^{\circ})$$
$$= 2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$
$$= 2\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2} - \sqrt{2}i$$

The required rectangular form is $\sqrt{2} - \sqrt{2}i$.

P3:

If $z_1 = 5(\cos 170^{\circ} + i \sin 170^{\circ})$ and $z_2 = (\cos 55^{\circ} + i \sin 55^{\circ})$, then find $z_1 . z_2$.

Solution:

Given, $z_1 = 5(\cos 170^\circ + i \sin 170^\circ)$

and $z_2 = \left(\cos 55^\circ + i \sin 55^\circ\right)$

Since, the modulus of the product of two complex numbers is the product of their moduli, and the amplitude of the product is the sum of their amplitudes.

$\therefore$ The modulus of $z_1 . z_2$ is $5 \times 1 = 5$

and amplitude $= 170^\circ + 55^\circ = 225^\circ$

Therefore, $z_1 . z_2 = 5(\cos 225^\circ + i \sin 225^\circ)$.

P4:

If $z_1 = 6(\cos 230° + i \sin 230°)$ and

$z_2 = 3(\cos 75° - i \sin 75°)$, then find $z_1 \div \bar{z}_2$.

Solution:

Given, $z_1 = 6(\cos 230° + i \sin 230°)$

$$z_2 = 3(\cos 75° - i \sin 75°)$$

$$\therefore \bar{z}_2 = 3(\cos 75° + i \sin 75°)$$

Since the modulus of quotient of two complex numbers is the modulus of the dividend divided by the modulus of the divisor, and the amplitude of the quotient is the amplitude of the dividend minus the amplitude of the divisor.

$\therefore$ The modulus of the quotient $\frac{z_1}{\bar{z}_2}$ is $\frac{6}{3} = 2$

And amplitude $=$ amplitude of z_1 $-$ amplitude of $\bar{z}_2$

$$= 230° - 75° = 155°$$

Therefore, $\frac{z_1}{\bar{z}_2} = 2(\cos 155° + i \sin 155°)$

IP1:

Find the polar form of $-1 - i\sqrt{3}$.

Solution:

Let $z = -1 - i\sqrt{3} = r(cos\theta + isin\theta)$, where

$$r = |z| = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = 2 ,$$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \tan^{-1}(\sqrt{3})$.

Notice that z lies in III-quadrant. So, $\theta = \frac{-2\pi}{3}$

$\therefore$ The Polar form of $-1 - i\sqrt{3}$ is $2(cos(-\frac{2\pi}{3}) + isin(-\frac{2\pi}{3}))$.

IP2:

Find the rectangular form of $2(\cos 225° + i \sin 225°)$.

Solution:

$$2(\cos 225° + i \sin 225°)$$
$$= 2(\cos(180° + 45°) + i \sin(180° + 45°))$$
$$= 2(-\cos 45° - i \sin 45°) = 2\left(\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = -\sqrt{2} - i\sqrt{2}$$

The required rectangular form is $-\sqrt{2} - \sqrt{2}i$.

IP3:

If $z_1 = 10(\cos 305^\circ + i \sin 305^\circ)$ and

$z_2 = 2(\cos 65^\circ + i \sin 65^\circ)$, then find $z_1 . z_2$.

Solution:

Given, $z_1 = 10(\cos 305^\circ + i \sin 305^\circ)$

and $z_2 = 2(\cos 65^\circ + i \sin 65^\circ)$

Since, the modulus of the product of two complex numbers is the product of their moduli, and the amplitude of the product is the sum of their amplitudes.

$\therefore$ The modulus of $z_1 . z_2$ is $10 \times 2 = 20$

and amplitude $= 305^\circ + 65^\circ = 370^\circ$

But we use the smallest positive coterminal angle

i.e., $370^\circ - 360^\circ = 10^\circ$

Therefore, $z_1 . z_2 = 20(\cos 10^\circ + i \sin 10^\circ)$.

IP4:

If $z_1 = 6(\cos 110° + i \sin 110°)$ and
$z_2 = \frac{1}{2}(\cos 212° + i \sin 212°)$, then find $\frac{z_1}{z_2}$.

Solution:

Given, $z_1 = 6(\cos 110° + i \sin 110°)$

and $z_2 = \frac{1}{2}(\cos 212° + i \sin 212°)$

Since the modulus of the quotient of two complex numbers is the modulus of the dividend divided by the modulus of the divisor, and the amplitude of the quotient is the amplitude of the dividend minus the amplitude of the divisor.

∴ Modulus of the quotient $\frac{z_1}{z_2} = \frac{6}{\frac{1}{2}} = 12$

and amplitude $= 110° - 212° = -102°$

But, we use the smallest positive coterminal angle

i.e., $360° - 102° = 258°$

$$\therefore \frac{z_1}{z_2} = 12(\cos 258° + i \sin 258°)$$

Exercises:

1. Express each of the following complex numbers in polar form:
   a. $-1 + i\sqrt{3}$
   b. $6\sqrt{3} + 6i$
   c. $2 - 2i$
   d. $-3 + 0i$
   e. $0 + 4i$
   f. $-3 - 4i$

2. Express each of the following complex numbers in rectangular form:

a. $4(\cos 240^{\circ} + i \sin 240^{\circ})$

b. $3(\cos 90^{\circ} + i \sin 90^{\circ})$

c. $5(\cos 128^{\circ} + i \sin 128^{\circ})$

3. Perform the indicated operations, giving the result in both polar and rectangular form.

a. $2(\cos 50^\circ + i \sin 50^\circ) \cdot 3(\cos 40^\circ + i \sin 40^\circ)$

b. $6(\cos 110^\circ + i \sin 110^\circ) \cdot \frac{1}{2}(\cos 212^\circ + i \sin 212^\circ)$

c. $10(\cos 305^\circ + i \sin 305^\circ) \div 2(\cos 65^\circ + i \sin 65^\circ)$

d. $4(\cos 220^\circ + i \sin 220^\circ) \div 2(\cos 40^\circ + i \sin 40^\circ)$

e. $6(\cos 230^\circ + i \sin 230^\circ) \div 3(\cos 75^\circ + i \sin 75^\circ)$

4. Express each of the numbers in polar form, perform the indicated operation, and give the result in rectangular form.

a. $\left(-1 + i\sqrt{3}\right)\left(\sqrt{3} + i\right)$

b. $\left(1 + i\sqrt{3}\right)\left(1 + i\sqrt{3}\right)$

c. $\left(3 - 3\sqrt{3}i\right)\left(-2 - 2\sqrt{3}i\right)$

d. $\left(4 - 4\sqrt{3}i\right) \div \left(-2\sqrt{3} + 2i\right)$

e. $-2 \div \left(-\sqrt{3} + i\right)$

f. $6i \div (-3 - 3i)$